November 15, 2001

Estimating Beta

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Abstract. This paper presents evidence that Ordinary Least Squares estimators of beta coefficients of major firms and portfolios are highly sensitive to observations of extremes in market index returns. This sensitivity is rooted in the inconsistency of the quadratic loss function in financial theory. By introducing considerations of risk aversion into the estimation procedure using alternative estimators derived from Gini measures of variability one can overcome this lack of robustness and improve the reliability of the results.

Key words: OLS estimators, systematic risk, mean-Gini

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1. Introduction

The valuation of risky assets is one of the major research tasks in financial economics that has led to the development of several Capital Asset Pricing Models, the most popular of which is the Sharpe-Lintner-Black mean-variance *CAPM*. In this model, the typical measure of asset riskiness is the beta, i.e. the covariance between the asset return and the market portfolio return. The basic tenet of *CAPM* lies in the separation of estimating beta risk from its pricing. Indeed *CAPM* assumes that one can define and measure systematic risk irrespective of risk aversion, which affects only the equilibrium pricing of individual assets. As is well known, this separation is valid only under the restrictive assumption of two-factor separating distributions or alternatively, if the utility function is quadratic.

Empirical asset-pricing models attract massive attention in finance, their goal being to assert or refute whether *CAPM* holds true. The traditional technique used to estimate the risk-expected return relation consists of two stages. In the first pass, betas are estimated from a time-series. In the second pass, the relationship between mean returns and betas is tested across firms or portfolios. This methodology has been the subject of much criticism that has led to many attempts at improvement. Such studies were initiated by Fama and MacBeth (1973) who introduced a rolling technique, and were followed by proponents of maximum likelihood estimation, for example Gibbons (1982), Stambaugh (1982), and Shanken (1992), to name a few. MacKinlay and Richardson (1991) developed a test for mean-variance efficiency without assuming normally distributed asset returns. However, *CAPM* suffered a major setback due to a series of papers published by Fama and French (1992, 1993, 1995, 1996, and 1997) who claimed that beta itself is not sufficient for explaining expected return. On the other hand, using alternative econometric and experimental techniques, Amihud, Christensen, and Mendelson (1992), Jagannathan and Wang (1996), and Levy (1997) rejected Fama and French results and reclaimed beta as the valid measure of risk in asset pricing . All these findings point to a major question: Is beta relevant in finance or is it merely mis-estimated?

Since its inception in finance, beta has been used mainly for two purposes. The first involves the ranking of assets and portfolios with respect to systematic risk by practitioners. The second deals with testing *CAPM* and mean-variance efficiency. The latter process involves a second stage regression (cross-section regression) intended for testing the efficiency of the market portfolio and the linear relationship between expected returns and betas, as discussed for example by Kandel and Stambaugh (1995).

An additional issue that complicates the problem of estimating beta is that one cannot separate the issue of risk aversion from the statistical loss function used in the estimation. As will be argued later, risk aversion signifies the asymmetric treatment of deviations from the regression of stock returns on market returns. On the other hand, statistical theory implies the equal treatment of observations. The clash between financial and statistical theories complicates the estimation procedure, and therefore, we restrict our study to estimating assets' riskiness and delay the pricing of risk to further research.

In this paper we question whether the standard procedure for estimating systematic risk is compatible with financial theory and show how the regression technique used to estimate systematic risk is not robust with respect to wide market fluctuations. The sensitivity of beta to the presence of extreme observations can give rise to data mining and lead the way to peculiar relationships.

We argue that beta sensitivity can be traced to a combination of two factors:

(i) Incompatibility between standard statistical methods and financial theory. In particular, the Ordinary Least-Squares (*OLS*) regression estimator is based on a quadratic weighting scheme that tends to contravene the assumptions of risk aversion;

(ii) Probability distribution of market returns with "fat" tails; that is the data do not follow a normal distribution.[1]

Accordingly, these factors make beta sensitive to market fluctuations and therefore *OLS* is inappropriate for estimating betas.

We suggest alternative estimators for beta that are robust with respect to extreme fluctuations in the market return. In this sense, we follow Chan and Lakonishok (1992) and Knez and Ready (1997) for the use of robust estimation procedures, but with a different rationale. In using trimmed regressions to seek robustness, crucial information regarding the behavior of securities returns with respect to market portfolio is removed for the sake of robustness. The data that becomes deleted may be considered by some investors as the most valuable because it represents information about the state of nature that concerns them the most. Therefore, we do not seek robustness by using statistical methods that are less sensitive to generally wide fluctuations. Rather, we seek to identify, according to economic theory, the relative weights that should be attached to different fluctuations. Adjusting the weighting scheme following economic theory allows for improving the systematic risk estimator at low cost.

To document the magnitude of the sensitivity of beta to market fluctuations and to avoid any influence of small or unusual companies, we first consider the 30 firms in the Dow Jones Industrial Average (*DJIA*) and then, 20 portfolios that have been built with the 100 largest traded firms. We use *CRSP* daily returns for a period of ten years (January 1984 through December 1993) making a total of 2528 observations. The use of the daily returns was guided by two motivations:

(i) Daily returns provide a relatively large amount of data. Since we are interested in the effect of extreme observations on the estimates, such as those that appeared during the 1987 Crash, a large amount of data will reduce the effect of any one observation. Ignoring the 1987 Crash altogether would imply that periods of high volatility do not play an important role in any estimate;

(ii) Monthly returns are obtained by averaging daily returns. Thus they are expected to deviate from normality less than daily returns do.[2] But as pointed out by Levy and Schwartz (1997), the longer the measurement period the lower will be the observed correlations among asset returns. Also, Levhari and Levy (1977) discuss the effect of the investment horizon on the empirical testing of beta. Hence, increasing the measurement period may "normalize" the distributions but at the same time will reduce the correlation among the returns which defeats the main goal in estimating beta.

Using these data, we conduct two experiments. In the first, the highest four market performance observations based on the *S&P 500* Index are removed from the sample and the betas are re-estimated. In the second, the highest four and the lowest four observations of the market are deleted (a total of less than 0.3 percent of the entire number of observations). Table 1a reports the deviations of the new estimates of betas in terms of their standard errors.[3]

When the highest four and the lowest four market returns are removed, the betas of 7 firms (out of 30) change by more than 4 standard errors. Moreover, the betas of more than 75% of the firms change by more than one standard error. When only the highest four observations are omitted, then the betas of 9 firms (30 percent of the firms) change by more than one standard error. However, the impression one gets from the standard errors using the entire sample, as reported in the Appendix, is that the probability of such occurrences is zero.

In Table 1b, these same conclusions are also obtained with beta ranked portfolios that reduce the problem of sharp return fluctuations for individual firms. When the highest four and lowest four market returns are omitted, the betas of 7 portfolios (out of 20) change by more than 3 standard errors. When the highest four observations are deleted, the betas of 9 portfolios change by more than one standard error, confirming the results for individual stocks.

These results indicate beta’s great sensitivity to extreme market fluctuations, which casts doubts on the robustness of the *CAPM* as it varies with respect to the choice of the sample period and the specification of the model. This sensitivity exists for both upward and downward movements of the market. Sensitivity to extreme downturn market fluctuations can be justified by arguing extreme cases of risk aversion, but it is not easy to explain sensitivity to extreme upward market movement.

The aim of this paper is to explore those factors that contribute to the sensitivity of beta estimates to extreme observations of market returns and to suggest alternative estimators that are both robust and better represent investors' risk aversion. In particular, the analysis shows that all models that represent the investor as an expected utility maximizer characterize systematic risk by a covariance formula between the marginal utility of wealth and the asset's return. The differences among the various models have to do with the exact specification of the marginal utility of wealth. Since this covariance cannot be observed, valuation models identify risk by a specific measure of variability, like the variance or the semivariance, and for the latent covariance substitute a covariance between the market portfolio and an asset's return. On the other hand, it is shown that the regression technique used to estimate systematic risk implicitly specifies the functional form of the marginal utility of income, and thus determines the implied risk aversion.

The paper is organized as follows. Section 2 presents the *OLS* estimator for beta as a weighted average of the change in asset return conditional on the change in market returns. The weights used in averaging depend solely on the distribution of market returns. As the weights are sensitive to extreme market fluctuations, the *OLS* estimation procedure attaches greater weights to extreme market changes, a characteristic that may contradict financial theory.

In Section 3, we show how financial theory implies that systematic risk is expressed by a covariance formula between the marginal utility of wealth and asset return. Provided that the utility function is known, this covariance defines the *ideal* beta, that is the beta that would be most suitable for reflecting the riskiness of the asset. Having defined the *ideal* beta enables us to compare alternative estimators of beta with the *ideal* beta. This comparison reveals that while it may be justified to attach greater weights to extreme downturn realizations of the market, it contradicts financial theory to attach greater weights to upward movements of the market. The property of attaching heavy weights to extreme positive high returns on the market portfolio challenges financial theory and simultaneously decreases the robustness of the estimator of beta.

In Section 4, we offer alternative estimators for describing the riskiness of an asset such as the extended Gini estimators, and investigate their properties. These estimators attach lower weights than the *OLS* estimators to upward market movements, thus making the estimator both more appropriate from the theoretical point of view, and at the same time more robust than the *OLS* estimator. Section 5 concludes the paper

2. The *OLS* estimator for beta

We introduce the *OLS* estimator of beta as a weighted average of the slopes of the lines delineated by two adjacent observations along the *Security Characteristic Curve*, (defined later). This enables us to show that *OLS* attaches too much weight to extreme observations of market return in the sample.

We consider a market model where security returns are continuously random and have a joint density function $f(R_k,M)$, where R_k is the return on asset k and M is the market portfolio. Let f_M, F_M, μ_M, and σ^2_M denote the marginal density, the marginal cumulative distribution, the expected value and the variance of M. We assume the existence of the first and second moments and define $R_k(m) = E(R_k | M = m)$ as the conditional expected return on asset k given the portfolio's return $M = m$. $R_k(m)$ is known as the security characteristic line (Sharpe, 1981), but here we refer to it as the *Security Characteristic Curve* since we do not assume a specific curvature.

In order to estimate the beta of the asset, it is usually assumed that the following relationship holds:

$$R_k = \alpha_k + \beta_k M + \varepsilon_k , \tag{1}$$

with the usual assumption that ε_k's are i.i.d. random variables with zero expected value and constant variance. The *OLS* estimator is then:

$$\beta_{OLS} = \frac{cov(R_k , M)}{cov(M , M)} . \tag{2}$$

where the index k is omitted.[4] Theorem 1 presents the *OLS* estimator of β_{OLS} as the weighted average of the slopes of the *Security Characteristic Curve*.

Theorem 1: Let $E^*(R_k|M) = \alpha + \beta M$ represent the best linear predictor of R_k, given M, and let $\delta_k(m)$ represent the derivative of the *Security Characteristic Curve* $R_k(m)$ with respect to m. That is, $\delta_k(m) = R_k'(m) = \partial E[R_k \mid M = m] / \partial m$, is the slope of the *Security Characteristic Curve* of security k when the market return equals m. Then β_{OLS} is the weighted average of the slopes of the regression curve:

$$\beta_{OLS} = \int_M w(m)\delta_k(m)dm, \tag{3}$$

where $w(m) > 0$, $\int w(m)dm = 1$ and the weights for *OLS* are given as:

$$\begin{aligned} w(m) &= \frac{1}{\sigma_M^2}[\mu_M F_M(m) - \int_{-\infty}^{m} t f_M(t)dt] \\ &= \frac{F_M(m)}{\sigma_M^2}[\mu_M - E(M|M \le m)] \ . \end{aligned} \tag{4}$$

Proof: See Yitzhaki (1996).

Theorem 1 presents the *OLS* estimator of β as a weighted average of the changes in the asset expected return conditional on the changes in the market return. The sum of weights is equal to one and is normalized by the variance of the market return. This means that the weighting scheme is actually the relative contribution of each segment of market return to the variance of market return. The second part of Equation (4) reveals that this contribution is based on the rank of the market return, $F_M(m)$, as well as to the expected contribution to mean market return of all the returns that are smaller than m.[5]

To understand the properties of the weighting scheme, we assume a specific distribution for the market return. We consider, in particular, the uniform distribution and the normal distribution, the first because of its simplicity, and the second because it is widely used to describe the distribution of market return.

A. *The uniform distribution*. Let M be uniformly distributed between *a* and *b*. By applying Equation (4), the weight attached to the slopes of the *Security Characteristic Curve* at *m* is:

$$w(m) = \frac{6(b-m)(m-a)}{(b-a)^3} . \tag{5}$$

This weighting scheme attaches the maximum weight to the midpoint of the market return. It is symmetric around the midpoint. Also, the farther the actual return is from the expected return, the lower is the weight.

B. *The normal distribution***.** Let M be a standard normally distributed variate. Then

$$w(m) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m} t e^{-t^2/2} dt = \frac{1}{\sqrt{2\pi}} e^{-m^2/2} . \tag{6}$$

The weight is identical to the density of the normal distribution. Hence, equal percentiles of the distribution receive equal weights.

These examples reveal that the *OLS* weighting scheme is determined solely by the distribution of the market return and is sensitive to the shape of the distribution. For a flat density function, such as the uniform distribution, the *OLS* estimator attaches greater weights to observations that are near the mean of the distribution. When the distribution is normal, an equal number of observations receive exactly the same weight.

Empirical evidence on the distribution of the market rate of return by Fama (1965) and by Mantegna and Stanley (1995) indicates fatter tails than expected from a normal distribution. Hence, the weighting scheme of the normal distribution is not a good approximation of the actual weighting scheme in a typical estimation of β. Therefore, instead of identifying the distribution type and evaluating the weighting scheme for given theoretical distributions, it is useful to establish the weights directly from the sample as follows.

Let us consider a sample of n observations of stock return r_i ($i = 1, ..., n$) and market return m_i. We arrange the observations by ranking them in ascending order of market return. We define

$\Delta_i = m_{i+1} - m_i > 0$ as the difference in stock returns, and $b_i = \frac{r_{i+1} - r_i}{m_{i+1} - m_i}$ ($i = 1, ..., n-1$) as the slope of

the line joining two adjacent observations. [6]

Theorem 2 : Within the sample, the *OLS* estimator of systematic risk β is the weighted average of slopes delineated by adjacent observations. That is,

$$b_{OLS} = \sum_{i=1}^{n-1} w_i b_i \tag{7}$$

where $w_i > 0, \Sigma w_i = 1$.

The weights are given by

$$w_i = \frac{\Delta_i \{ \sum_{j=i}^{n-1} i(n-j)\Delta_j + \sum_{j=1}^{i-1} j(n-i)\Delta_j \}}{\sum_{k=1}^{n-1} \Delta_k \{ \sum_{j=k}^{n-1} k(n-j)\Delta_j + \sum_{j=1}^{k-1} j(n-k)\Delta_j \}} \; . \tag{8}$$

Proof: See Yitzhaki (1996).

The components of Equation (7) are the slopes b_i and the weights w_i that depend solely on the distribution of the independent variable, i.e., the market return. The contribution of each observation to the estimator of β consists of (i) the effect of the weighting scheme and (ii) the slope of the line joining two adjacent observations on the empirical *Security Characteristic Curve*. Adding up the weights of a range of market returns yields the contribution of that range of the distribution of the market return to the estimate of β.

Equation (8) reveals that the weights w_i depend on both the rank of the observation of market return and its distance from the adjacent observation, as defined by Δ_i. The weight increases as the rank of the observation draws closer to the median and as the distance between observations increases. That is: (i) the *OLS* weighting scheme attaches more weight to observations that are closer to the median of the market return, and (ii) the weight is an increasing (quadratic) function of the distance between adjacent market observations.

Under the uniform distribution, the distances between observations are equal, hence the only factor determining the weight is the difference between the rank of the market return from the median of the distribution. Yet, if the distribution is normal, the farther the observation is from the median the greater will be the distance between observations. The result is that these two factors offset each other, leading to a weighting scheme that attaches equal weights to an equal number of observations.

If the distribution has fatter tails than those of a normal distribution, the distances between adjacent observations increase at a faster rate (than the normal distribution rate) as one moves away from the center. In this case, the weighting scheme assigns greater weights the farther the observation is from the median. This property causes the *OLS* to be sensitive to extreme observations.

Table 2 presents the weighting scheme for our sample together with some additional statistics. The *S&P 500* returns are aggregated into 10 groups, each with ten percent of the observations (deciles), from the lowest (Decile # 1) to the highest (Decile # 10). In other words, the returns on the *S&P 500* Index are ranked from a "bear" market to a "bull" market during the sample period. The entries in Table 2 serve to shed some light on the effect of the state of the market on beta.

The lowest observation in the sample is a daily return of minus 20 percent, and the largest one is a return of plus 9 percent, so that the range of the market return in the sample period is around 30 percentage points.

The first column in Table 2 reports the number of observations in each decile. The second column reports the percentage of the range of the market return used by each decile. The wider the range of the deciles the lower the density function at that segment of the range. The first decile covers 66 percent of the range while the highest decile covers 27 percent. This implies that the sample distribution has fat tails in that 20 percent of the observations cover 93 percent of the range. Also, there are more extreme observations at the lower tail than at the upper tail.

The third column presents the contribution of each decile to the variance of the *S&P 500* return. Formally:

$$S_n = \frac{10}{n\,\sigma_M^2} \sum_{i \in D_n} [\,m_i - E(m)\,]^2 \tag{9}$$

where S_n is the contribution of the n_{th} decile to the variance, and D_n represents the n_{th} decile. That is, the contribution of each observation to the variance is aggregated for each decile to obtain the contribution of the decile to the variance of the market.[7] The contribution of the lowest decile to the variance is 50 percent, while the highest decile contributes 32 percent. Taken together, both deciles contribute 82 percent to the variance of the *S&P 500* return.

The fifth column shows the contribution of each decile to the *OLS* weighting scheme, according to Equation (8). The slopes of the *Security Characteristic Curve* are multiplied by this weighting scheme to obtain the betas for the individual firms. The first decile is assigned 34 percent of the weight, while the upper decile is assigned 16 percent. Taken together, the first and the last decile are responsible for fifty percent of the weights. If the *S&P 500* Index return were to be normally distributed, we should find the weight attached to the two extreme deciles to be 20 percent; If the distribution were to be uniform, then the weight attached to the two extreme deciles should be less than 6 percent. The conclusion is that the fat tails are responsible for the instability of the *OLS* estimator of systematic risk.

The remaining entries for the column show that, excluding the extreme deciles, the distribution is symmetric around the fifth and six deciles. Indeed, the weight attached to the fifth decile is identical to the weight attached to the sixth decile; the weights attached to the fourth and seventh deciles are almost identical and so on. (The last two columns present the weights used for the alternative estimators discussed in Section 3).

Table 2 reveals that one factor contributing to the instability of betas lies in the larger weight allocated to extreme observations. The *OLS* attaches 50 percent of the weight to 20 percent of the observations. The evidence that a relatively high number of *DJIA* stocks are sensitive to the exclusion of some observations hints that the *Security Characteristic Curve* may be non-linear, although this issue is beyond the scope of the current paper since *CAPM* does not require linearity of the *Security Characteristic Curve*. As is argued in the next section, portfolio theory and *CAPM* do not require linearity of the market line nor the use of a regression model to compute beta.[8] Indeed, under the original *CAPM*, beta is an *ex-ante* measure of risk. The use of regression analysis to estimate beta confuses the issue because one has to assume error terms that are uncorrelated with the market portfolio. (Sharpe, 1991).

For our purposes, we distinguish between sensitivity to the upper tail (a "bull" market) and the lower tail (a "bear" market) of the distribution. The greater weight attached to the lower tail of the market distribution can be explained by high levels of risk aversion. Indeed, a risk-averse investor attaches greater weight to the lower tail of the distribution. Therefore, the higher the risk aversion, the greater the weight that should be attached to the lower portion of the distribution. In the extreme case of a max-min investor, all the weight is attached to the lowest two observations of the market, in order to determine the contribution of the asset to the riskiness of the portfolio (Shalit and Yitzhaki, 1984).

While risk aversion can justify the greater weight assigned to the lower tail, it is not at all clear what would justify the large weight attached to the higher decile of the distribution. On the contrary, risk aversion would indicate that weights be reduced in the face of a bull market. Furthermore, in the extreme case of risk neutrality, the weights should not exceed the weight that is attached when calculating the mean of the distribution. Hence it seems difficult to reconcile the quadratic weighting scheme implied by *OLS* with a weight that increases with market returns on the one hand and with risk aversion on the other.

In some sense the number of observations is misleading. Although we have a sample of 2528 observations, only 20 percent of these are actually responsible for 50 percent of the coefficient. Because the weights are not evenly distributed in each decile, the effect is that we obtain the sensitivity of beta via the extreme observations.

The issue of the weighting scheme pervades other statistics of the regression such as mis-specification tests and tests based on the error term distribution. Since mis-specification tests rely on the same weighting scheme, there is no guarantee that such tests do not rely on estimators that stress the non-relevant portions of the distribution. Testing for undue influence may therefore be helpful. The effect of an observation takes two forms: (i) its weight and (ii) the deviation of its slope (defined with adjacent observations) from the average slope. If only observations with low weights deviate from the average slope, the influence of each observation may be minor, either because of the low weight or because of the small deviation of the slope.

We now analyze the implication of economic theory on the estimation procedure of systematic risk.

3. Systematic risk

Although beta lost some of its glitter as a result of some empirical tests (e.g., Fama and French (1992)), it still maintains its theoretical appeal as a measure of risk. To stress the importance of systematic risk in financial theory, we argue that only two parameters, the expected return and the systematic risk, are sufficient in order to capture the entire effect of the distribution of the asset's rate of return on the expected utility of the investor. Systematic risk is expressed as the covariance between asset return and the marginal utility of wealth. However, the marginal utility of wealth is implicitly assumed. It is the estimation procedure of systematic risk that dictates the implied expected marginal utility of wealth. *OLS* implies a quadratic utility which means that the marginal utility of wealth is a linear function of wealth. This assumption of linearity is responsible for the sensitivity of beta to fat tails.

The *CAPM* relating beta to expected return was originally derived by assuming mean-variance efficiency of portfolios.[9] Afterward, following the Rothschild and Stiglitz (1970) increasing risk model, Merton (1982, 1990) proved the *CAPM* relationship for all risk-averse investors by postulating efficient portfolios, in the sense that investors maximize expected utility. Portfolio efficiency, whether in mean-variance or expected utility, is the condition necessary to derive *CAPM*s. As Roll (1977) noted however, market efficiency is itself the stipulation that prevents a valid test of asset pricing theory.

We alleviate this problem by showing how the expected utility or disutility produced by a small change in the holdings of one asset in the portfolio can be captured in terms of expected return and systematic risk. This result is valid for all expected utility maximizers, and hence depends neither upon the quadraticity of preferences nor the distribution of returns.

We do not assume that investors actually succeed in maximizing expected utility, thus departing from Merton (1982, 1990). Although the intended goal is to maximize expected utility, it may not be achieved because of transaction costs, a lack of perfect information, constant changing environments, or other commitments. Hence, the definition of systematic risk does not depend on whether investors are able to maximize the expected utility since it is sufficient to assume that they intend to do so.

Consider an expected utility maximizing investor who holds a mixed portfolio of risky and safe assets. The investor's goal is represented by the maximization problem:

$$\begin{aligned} &\underset{\alpha_1,\ldots,\alpha_N}{Max} \quad E[U(M)] \\ &s.t. \quad M = M_0 [y + \sum_{i=1}^{n} \alpha_i R_i] \\ &\qquad \sum_{1=1}^{n} \alpha_i = 1 \text{ and } M_0 \equiv 1 \quad , \end{aligned} \tag{10}$$

where $E[U(M)]$ is expected utility; U is a continuous, monotonically increasing concave function, M_0 is a given initial wealth assumed to be 1 without loss of generality; α_i and R_i are, respectively, the share of M invested in asset i and the returns on asset i; and y is the return on some other income, either deterministic or stochastic that can be attributed to labor or human capital.

The investor holds a given portfolio $\{ \alpha^0 \}$, whose shares are α^0_i, $i = 1, ..., n$. Note that the only requirement on α^0 is that it is held by the investor. Assume the investor wants to change the holdings of asset k in the portfolio. The effect of increasing α^o_k on expected utility is given by:

$$\frac{\partial E[U(M)]}{\partial \alpha_k} = E[U'(M) R_k] - \lambda \quad , \tag{11}$$

where λ is the Lagrange multiplier associated with the portfolio constraint. By adding and subtracting $E[U'(M)]\mu_k$, where μ_k is the expected return on asset k, we can rewrite Equation (2) as:

$$\frac{\partial E[U(M)]}{\partial \alpha_k} = E[U'(M)]\mu_k + cov[U'(M), R_k] - \lambda \quad . \tag{12}$$

Our purpose here is to compare between assets. Hence, all factors that are equal for all assets can be ignored. Equation (12) expresses the effect of a marginal increase in asset k on expected utility as a function of the expected return on asset k, and the asset's systematic risk, defined as the covariance between marginal utility of wealth and the return on asset k.

Assuming a specific utility function enables us to obtain an explicit expression for systematic risk. For example, a quadratic utility function defined on wealth leads to the conventional beta. Formally, let us denote $U(M) = A + B\,M + 0.5\,C\,M^2$; then $Cov\,[\,U'(M)\,,\,R_k\,] = \beta_k\,\sigma^2_M$, where $\beta_k = \beta_{OLS} = Cov(R_k,M)/cov(M,M)$ is the *OLS* regression coefficient of R_k on M. If the utility function is defined in terms of the rate of return on the portfolio rather than the level of wealth, the standard expression for systematic risk is produced.

If investors identify risk with another index of variability, such as the semivariance or the Gini's Mean Difference (*GMD*), alternative expressions for beta may be obtained (Shalit and Yitzhaki (1984)). In the case of *GMD*, $\beta_k = cov[R_k,F(M)]/cov[M, F(M)]$ where $F(M)$ is the cumulative probability distribution of investors' wealth. The implied utility function in this case can be traced to a special case of Yaari's (1987) dual approach to risk aversion.

An alternative to the assumption, explicit or implicit, of a specific utility function is to consider a set of utility functions that comply with second degree stochastic dominance (*SSD*), where the set of eligible functions is composed of all utility functions with $U'() > 0$ and $U''() < 0$. [10] Then one can summarize the effect of an increase in the share of an asset on all possible legitimate utility functions by a curve instead of a single parameter as in the case of specific utilities.

Under *SSD*, the effect of increasing α^0_k on expected utility is captured by the following Theorem:

Theorem 3: A necessary and sufficient condition for a small increase in asset *k* to increase expected utility for all functions with $U' > 0$ and $U'' < 0$ given portfolio *M* is

$$ACC_k^M(p) = \int_{-\infty}^{m} R_k(t) f_M(t)\,dt \ \geq\ 0 \quad \text{for all } m, \text{ where } \quad p = \int_{-\infty}^{m} f_M(t)\,dt \quad , \tag{13}$$

where *ACC* stands for the absolute concentration curve and $R_k()$ is the conditional expected return on asset *k* given a portfolio return.

Proof: See Shalit and Yitzhaki (1994), where the theorem is proved for an increase in the share of one asset subject to a decrease in the share of another asset.

The *ACC* of asset *k* sums up the conditional expected returns on asset *k*, each weighted by the probability of the portfolio $M = m$. The probability p is that the portfolio return is at most *m*. Hence for a given probability *p*, the *ACC* is the cumulative expected return on asset *k*, subject to a portfolio return of, at most, *m*.

By adding and subtracting $p\mu_k$ one can derive the necessary conditions of Theorem 3 in terms of expected return and beta as follows:

$$ACC_k^m(p) = p \cdot \mu_k - \int_{-\infty}^{m} [\mu_k - R_k(t)] f_M(t)\, dt \geq 0 \text{ for all } m \text{, where } p = \int_{-\infty}^{m} f_M(t)\, dt \; . \tag{14}$$

Since Equation (14) holds for all *p*, it implies that:

$$\int_0^1 ACC_k^M(p)\, dp = \int_0^1 \mu_k p\, dp - \int_0^1 \int_{-\infty}^{m} [\mu_k - R_k(t)] f_M(t)\, dt\, dF(m) \quad , \tag{15}$$

which is written as

$$\int_0^1 ACC_k^M(p)\, dp = 1/2\, \mu_k - cov[R_k, F_M(M)] \geq 0 \quad . \tag{16}$$

Dividing and multiplying the covariance term by one-half of the *GMD* of the portfolio produces the beta of asset *k* obtained using the Gini as a measure of risk. [11] Hence, a necessary condition for a small increase in asset *k* to increase expected utility for all risk-averse investors given that they hold portfolio α^0 is that:

$$\mu_k - \Gamma_M \beta_k \geq 0 \tag{17}$$

where

$$\beta_k = \frac{cov(R_k, F_M)}{cov(M, F_M)} \tag{18}$$

The conclusion drawn in this section is that for all expected utility models, the effect of an increase in the holding of an asset on expected utility can be broken down into the effects of the expected return and the systematic risk (beta) of the asset. The systematic risk is the covariance between the marginal utility of wealth in the appropriate portfolio and the return on the asset. This covariance, normalized by an appropriate measure of variability, is also the regression coefficient of the asset return on the portfolio's return. Thus, the choice of the regression technique is also the choice of marginal utility of the wealth function, and thus of risk aversion.

Without restricting the set of distributions, the *OLS* regression method implies a quadratic utility function that is sensitive to extreme observations. Thus *OLS* may be inappropriate as it could imply negative marginal utility of wealth for extreme observations. Indeed financial theory under risk aversion assumes a declining marginal utility of wealth. Furthermore, the higher the degree of risk aversion, the faster marginal utility declines with wealth. This implies that risk-averse investors attribute less weights for variability when market returns are high than when market returns are low. On the other hand, as seen from Equation (4), since the *OLS* weighting scheme is symmetric with respect to the median of market returns, *OLS* will be incompatible with risk aversion and declining marginal utility of wealth.

The bottom line of our argument is that investors' risk aversion as it appears in *CAPM* must be the same risk aversion used when estimating systematic risk. In the next section, we propose an alternative estimation method of the market line that will be both more robust than the *OLS*, and will at the same time better reflect the risk aversion of the investors.

4. Alternative estimators for beta

Theorems 1 and 2 provide a formal explanation for the well-known observation that *OLS* estimators are sensitive to outliers. A common solution to this problem is to remove extreme observations from the sample. The inconsistency in this procedure is inherent. On the one hand, outliers receive extremely high weights. On the other hand, when they appear to have a strong effect on the estimate, they are arbitrarily deleted from the sample along with the relevant information they carry.

The *OLS* estimator can be interpreted as a weighted average of slopes. As we obviously do not want to change the slopes, the solution must rely on changing the weighting scheme and making it compatible with risk averse behavior. The weighting scheme depends upon (i) the rank of each observation and (ii) the difference (in terms of the independent variable) between each observation and the one adjacent. In *OLS* this difference is raised to the second power, thus exacerbating its effect.[12]

An alternative strategy is one that yields a weighting scheme that is consistent with one's perception of risk aversion. Therefore, one should use indices of variability that are less sensitive to extreme high rates of market return and that imply regression slope estimators with low weights to extreme high observations. Such properties exist in the Gini estimators.

As far as we know , Gini's statistics were first applied as measures of dispersion in financial data by Fisher and Lorie (1970) who justify their use ". . . because many of the distributions . . . depart greatly from normality. For such distributions, the standard deviation of even a large sample may not give a very meaningful indication of the dispersion of the population. Gini's mean difference and coefficient of concentration are nonparametric measures and are invulnerable to this consequence of departure from normality . . . Gini's mean difference differs from the mean deviation by giving greater weight to extreme observations, thus taking care of a frequently made criticism of the mean deviation. " (1970, p. 104)

The extended Gini is a family of variability measures that was applied by Yitzhaki (1983) to develop necessary conditions for stochastic dominance. It was used by Shalit and Yitzhaki (1984) to develop the Mean-Gini *CAPM* which is similar to *MV-CAPM* in its properties but, unlike *MV*, is consistent with expected utility for any concave function or any probability distribution. Even if one wants to rely on *MV-CAPM* as the theoretical basis for developing systematic risk, it may be helpful to use the beta Gini estimator as a robust estimate of beta.[13] That is to say , one can recommend the Gini beta on its own merits or as a robust estimator for the *MV* beta.

The extended Gini regression coefficient (*EGRC*) is a ratio based on the extended Gini variability index. The denominator denotes the market extended Gini index and the numerator the extended Gini covariance of the stock return with the market return.[14] *EGRC* is defined as: [15]

$$\beta(\nu) = \frac{cov(R, [1 - F_M(M)]^{\nu-1})}{cov(M, [1 - F_M(M)]^{\nu-1})} \quad \nu > 0 \quad , \nu \neq 1 \tag{19}$$

where ν is a parameter determined by the investigator to reflect the investigator's perception of risk aversion in the market.[16] If $\nu = 1$, the investigator assumes a risk-neutral market. The higher the ν the more risk-averse is the market. In the extreme case ($\nu = \infty$), the market concerns itself only with huge crashes (which is exhibited by max-min behavior). The range $0 \leq \nu < 1$ reflects risk-loving behavior with $\nu \to 0$ showing a max-max strategy; i.e., the market considers only the day with the highest return as the decision statistic. Another interpretation of ν is to view as a parameter in a specific set of utility functions belonging to the set of utility functions that comply with Yaari's (1987) dual approach to decision under risk.

The weighting scheme of the estimator for $\beta(\nu)$ is determined by the parameter ν and by the distribution of the market return. By determining ν, the investigator introduces individual perception of risk aversion into the estimation procedure.

Theorem 4: The extended Gini estimators of the regression coefficient have the following properties:

(a) In the population the parameters are the weighted averages of the slopes of the *Security Characteristic Curve*:

$$\beta(\nu) = \int V(m, \nu)\, \delta(m)\, dm \ , \tag{20}$$

with $V(m,\nu) > 0$ and $\int V(m,\nu)dm = 1$, where

$$V(m, \nu) = \frac{[1 - F_M(m)] - [1 - F_M(m)]^{\nu}}{\int_{-\infty}^{\infty} \{[1 - F_M(t)] - [1 - F_M(t)]^{\nu}\}\, dt} \ . \tag{21}$$

(b) In the sample, all estimators are weighted averages of slopes defined by pairs of adjacent observations:

$$b(\nu) = \sum_{i=1}^{n-1} V_i(\nu)\, b_i \ , \tag{22}$$

where $b_i = \dfrac{r_{i+1} - r_i}{m_{i+1} - m_i}$ $(i = 1, ..., n-1)$; $V_i > 0$, $\Sigma V_i = 1$, and

$$V_i(\nu) = \frac{[n^{\nu-1}(n-i) - (n-i)^{\nu}]\Delta_i}{\sum_{k=1}^{n-1} [n^{\nu-1}(n-k) - (n-k)^{\nu}]\Delta_k} \ . \tag{23}$$

(c) The estimators $b(\nu)$ are ratios of *U*-statistics. Hence, they are consistent estimators of $\beta(\nu)$. For large samples, the distribution of the estimators converges to a normal distribution. Furthermore ,for integer ν, the estimators $b(\nu)$ are unbiased and have a minimum variance among all unbiased estimators.

(d) Suppose that $E(Y|X) = \alpha + \beta X$ and $Var(Y|X) = \sigma^2 < \infty$. Then all extended Gini estimators are consistent estimators of β.

Proof: See Yitzhaki (1996) and Schechtman and Yitzhaki (1998).

Properties (a) and (b) show that all *EGRC*(ν) are weighted averages of the slopes defined by adjacent sample points. The differences among the estimators are in the weighting schemes. Property (d) of Theorem 4 enables the use of *EGRC* as an estimator for *MV*-beta.

To estimate *EGRC* it is not necessary to assume a linear relationship. All that is required is to assume a regression curve and an interest in estimating a weighted average of the curve slopes. The weights are determined by two elements: The first is the perception of risk aversion and the second is the statistical property of the estimate that depends on the distribution of the market return.

For a given ν, we can ignore the denominator as a normalizing factor and consider the numerator as a function of the value $p = i/n$ provided by the cumulative distribution:

$$w(p) = c(\nu)[(1 - p) - (1 - p)^{\nu}], \tag{24}$$

where c is a function of ν. By looking at the derivatives of w with respect to the cumulative distribution p, we can trace the properties of the weighting scheme:

$$\frac{\partial w}{\partial p} = \frac{\partial w[F_M(m)]/\partial m}{\partial F_M(m)/\partial m} = c(\nu)[\nu(1 - p)^{\nu-1} - 1]$$

$$\text{and} \quad \frac{\partial^2 w}{\partial p^2} = c(\nu)\nu(\nu - 1)(1 - p)^{\nu-2} .$$

For $\nu > 1$, i.e., for a risk-averse investigator, the weighting scheme increases for low values of p, reaches a maximum, and then declines. If $\nu < 1$, i.e., for a risk-loving investigator, the weighting scheme increases with p.[17]

If $\nu = 2$, the denominator of Equation (19) is one-half the *GMD*, while the numerator is the Gini covariance. The weighting scheme is symmetric in p. The closer the observation is to the median, the greater is its weight.

The *GMD* weighting scheme is similar to the *OLS* weighting scheme, except that the *OLS* weighting scheme uses quadratic distances between observations, while the *GMD* weighting scheme uses absolute differences. If the distance between observations of the market return is constant, such as in the uniform distribution, the weighting scheme of *GMD* is identical to that of *OLS*. If the distribution of the market return is normal, *GMD* attaches higher weights to the center of the distribution than *OLS* does. That is, while the *OLS* attaches equal weights to equal numbers of observations, the *GMD* attaches higher weights to the middle of the distribution. In general, since the denominator in Equation (19) is the extended Gini, it can be shown that the *GMD* (and the extended Gini) weighting schemes attribute to each decile of the distribution the contribution of that decile to the *GMD* (or the extended Gini).

The weighting scheme of the *GMD* beta is shown in the fifth column of Table 2. It can be seen that the *GMD* attaches only 15 percent of the weight to the lowest decile and 12 percent to the highest decile. Thus, the use of the Gini reduces the weight attached to the two extreme deciles of the market return from 50 percent under *OLS* to 27 percent under *GMD*.

The effect of changing ν on the weighting scheme is more complex, since both the numerator and the denominator are affected. If $\nu > 1$, that is, if risk aversion is considered, higher weights are given to the lower segments of the market distribution returns.

The weighting scheme for each ν can be calculated numerically. The last column of Table 2 presents the weighting scheme for $\nu = 5$. By raising ν from 2 to 5, the weight assigned to the lowest decile increases to 26 percent, while the weight assigned to the highest decile decreases from 12 to 6 percent. This approach enables the investigator to control the weighting scheme and to adjust it to the market risk aversion.

Table 3 presents the sensitivity of beta derived under various approaches to the deletion of the top four and the bottom four market return observations. Since the standard error of each estimate is derived under different assumptions, the estimates are not comparable. Therefore, Table 3 reports the absolute deviation of beta in each method in percentage points of the estimate. The first four columns report the effect of deleting the top four observations. In this case, the β_{OLS} of more than six firms (20 percent of the sample) changes by more than 3 percentage points, but none of the Gini betas (with $\nu = 2, 4, 6$) change by this magnitude. The *OLS* betas of an additional eight firms change by more than 2 percent while only the *GMD* betas of two stocks have changed by this magnitude. As expected, the higher the ν the less sensitive is the estimate of beta to the firm performance in extreme high market returns. To illustrate, note that for $\nu = 6$ all betas have changed by less than 1 percent.

The second four columns report the sensitivity of the estimates when both the top and bottom four observations are omitted. Under *OLS*, the betas of more than eight firms change by more than 10 percent, while the Gini betas in no case change by that magnitude. It also can be seen that increasing ν increases the sensitivity of beta because of the increase in the weight given to bottom observations. Nevertheless, the Gini method continues to be more robust than *OLS*. If one continues to increase ν, however, this property disappears.

The extended Gini approach offers an infinite number of alternative estimators. Two immediate questions arise: should the Gini method be used as a substitute for *OLS* estimation and, if so which ν should be chosen. The answer is not clear-cut. On the one hand, it depends on the degree of confidence one has about the market's risk aversion and, on the other hand, on the curvature of the empirical *Security Characteristic Curve*. If the *Security Characteristic Curve* is linear, it does not matter which method is used. If the slopes of the curve differ, i. e., if the relationship between the firm and the market is not linear, the method used is important. In this sense, *EGRC* offers a statistical test on whether incorporating the parameter of risk aversion in the estimation is important. As shown by Gregory-Allen and Shalit (1999) statistically testing the equality of various $\beta(\nu)$ is actually a test on the linearity of the regression curve, and therefore can determine whether ν is significantly important.

Ignoring risk aversion and assuming the validity of *MV*, *OLS* is the most efficient estimator. If the model is linear, using a Gini-based estimator will result in loss of efficiency. The larger the sample size however the less important is the efficiency loss.

If the *Security Characteristic Curve* is not linear, and if the stock market exhibits great volatility, *OLS* may lead to estimating wrong coefficients. Hence *MV-CAPM* may fail, not because of wrong assumptions but rather because of the estimation procedure. Further research would be fruitful in this area.

The Gini method does not need to specify the curvature of the *Security Characteristic Curve*. It provides a weighted average of the slopes where the weights are determined by risk aversion (theory) and statistical (curvature) considerations.

5. Further research and conclusion

The question remains as to how a Gini-based model can serve as a basis for testing *CAPM*. This is a complex issue because of the infinite number of risk aversion parameters ν. Indeed one should ask what is the proper ν to be used? With *MV*, risk aversion differentiation among agents vanishes in market equilibrium because of the separation theorem. Differences in risk aversion among investors only materialize in the various combinations of the riskless asset and the market portfolio. This is not the case with *MEG* betas, where risk aversion consideration appears in the estimation of systematic risk. This issue was also studied by Levy (1978) within *MV*. By considering investors who hold portfolios of stocks whose number is smaller than the number of stocks in the market, Levy shows that several *MV* betas for the same stock co-exist in equilibrium.

In the same spirit, we posit that different betas generated from various risk aversion coefficients can coexist in equilibrium in the same market. The challenge would be to establish the theoretical foundations for such a model. In the meantime we are compelled to use a market equilibrium with a unique Gini beta based on one coefficient of risk aversion, ν. For that purpose, one approach is to estimate ν most likely held by investors by comparing the mean-Gini efficient sets with the market portfolio, as Shalit and Yitzhaki (1989) have done.

In all expected utility models, the systematic risk of an asset can be presented as a function of a covariance between marginal utility of wealth and the return on the asset. In estimating beta by *OLS*, the marginal utility is represented by a linear function of the market return, implying that the utility function is quadratic. Together with the fat tails of the market distribution, this leads to an unwarranted sensitivity of the beta and to contradiction of the assumption of positive marginal utility of wealth. Further research is needed to verify that the cross-section estimating *CAPM* will not yield negative marginal utility of wealth. The *GMD* beta does not suffer from this problem and offers an alternative and robust estimation method of systematic risk.

Acknowledgements

Our thanks to Eugene Kandel, Eli Elyasiani, and one anonymous referee for their helpful comments and suggestions.

Notes

1. Recently followed by Mantegna and Stanley (1995), Mandelbrot (1963) and Fama (1965) supply the early evidence that the distribution of stock returns is characterized by fat tails relative to a normal distribution. See also Chan and Lakonishok (1992) for an excellent survey of the literature on non-robustness in estimating betas.

2. See the results in Brown and Warner (1985) and Fama (1965) that daily returns depart from normality more than monthly returns.

3. The estimates for betas are presented in the Appendix.

4. The index *k* will be omitted throughout the paper to avoid confusion.

5. In the sample, the cumulative distribution $F_M(m)$ is estimated by the rank of *m*.

6. To simplify the presentation we assume without loss of generality that $\Delta_i > 0$. Otherwise, we would need to aggregate all observations with the same market return *m* , a procedure that complicates the presentation without adding insight.

7. Note that the numerators of both Equations (9) and (8) present alternative decompositions of the variance of the market return. However, while Equation (9) presents it the usual way by writing the variance as squared deviations from the mean, Equation (8) presents it using the change in market return, Δ_i, instead of the market return, *m*.

8. Roll's (1977) critique is mainly concerned with the linearity of the relationship between beta and expected return, which is beyond the scope of this paper.

9. The *CAPM* was first derived by Sharpe (1964), Lintner (1965), and Mossin (1966). Merton (1972) provides a rigorous mathematical analysis of the efficient frontier which also appears in Roll's (1977) widely cited paper.

10. For the definition of the *SSD* criterion see Hadar and Russell (1969) and Hanoch and Levy (1969). See Levy (1992) for a thorough survey of the methodology.

11. The *GMD* of portfolio *M* is defined as $\Gamma_M = 2cov(M, F_M)$.

12. The quadratic differences can be attributed to calculating the variance.

13. See Carroll, Thistle, and Wei (1992) for a discussion of the robustness properties of the beta Gini estimator.

14. For simplicity of exposition only the parameters are presented. The estimators take the same form as the parameters, except that sample statistics are used instead of population parameters, and the empirical distribution is used instead of the cumulative distribution. See Schechtman and Yitzhaki (1998) for an investigation of the large sample properties of the estimators in a multiple regression framework; Olkin and Yitzhaki (1992) for an investigation of large sample properties of the regression based on the *GMD*, and Yitzhaki (1991) for a derivation of the standard error of the estimators.)

15. This estimator can be interpreted within the framework of *OLS* regression as an instrumental variable estimator with instrument $[1-F_M(M)]^v$. This provides a solution in the presence of errors-in-variables problems, Durbin (1954).

16. Two different questions arise when a measure of variability is used to represent risk. The first is how risk aversion is defined, and the second is how much expected return one is ready to sacrifice in order to reduce exposure to risk. The first question is answered by the choice of the index of variability used (variance, semivariance, extended Gini), while the answer to the second question depends on the curvature of the efficient frontier of risk (measured by the appropriate index) as compared to expected return.

17. The risk aversion implied by the extended Gini coefficient can be described by a specific case. Consider the utility function: $U[\mu, \Gamma(\nu)] = \mu - \Gamma(\nu)$ where ν is equal to an integer and Γ(ν) is the extended Gini coefficient. Then it can be shown that $\mu - \Gamma(\nu) = E[\,Min\{M_1, M_2,...,M_\nu\}\,]$, where M_i are i. i. d. variables [Yitzhaki (1983)]. That is, the investor maximizes the expected value of the minimum of ν random draws from the market return. Hence, when ν equals one the investor does not care about risk; when ν converges to infinity the investor behaves as if the worst case scenario will certainly occur. A max-max investor behaves as if the best realization is always obtained.

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Table 1a: Number of *DJIA* 30 firms whose betas change according to magnitude of change

Magnitude of Change in β in Terms of Standard Error of the Estimate	Omitting 4 highest returns	Omitting 4 highest & 4 lowest returns
Total Number of Firms	30	30
Less than 1 Standard Error	21	7
More than 1 and Less than 2 Standard Errors	8	4
More than 2 and Less than 3 Standard Errors	1	8
More than 3 and Less than 4 Standard Errors	0	5
More than 4 Standard Errors	0	7

Table 1b: Number of portfolios whose betas change according to magnitude of change*

Magnitude of Change in β in Terms of Standard Error of the Estimate	Omitting 4 highest returns	Omitting 4 highest & 4 lowest returns
Total Number of Portfolios	20	20
Less than 1 Standard Error	11	6
More than 1 and Less than 2 Standard Errors	9	6
More than 2 and Less than 4 Standard Errors	0	7
More than 4 Standard Errors	0	1

* Twenty portfolios, each composed of 5 firms are built by ranking the largest 100 traded firms according to their betas. Detailed results are given in Table A2.

Table 2: The *OLS* and *GMD* weighting schemes

Decile	Number of Observations	Percent of Range	Percent of Variance	*OLS* Weights	*GMD* Weights	*GMD*(ν) Weights
1	253	66.1	50.3	34.1	15.0	26.4
2	253	1.3	5.6	7.9	9.9	14.7
3	253	0.8	2.0	6.4	9.2	11.8
4	253	0.6	0.6	5.3	8.4	9.1
5	253	0.5	0.1	4.9	8.0	7.5
6	253	0.5	0.1	4.9	7.9	6.5
7	253	0.6	0.6	5.6	8.8	6.2
8	253	0.9	2.1	6.8	9.9	6.1
9	253	1.4	6.4	8.3	10.6	5.8
10	252	27.1	32.2	15.7	12.2	6.0
Total	2529	100.0	100.0	100.0	100.0	100.0

Table3: Percentage deviation of beta *OLS* and beta Gini

	β_{OLS}	$\beta_{v=2}$	$\beta_{v=4}$	$\beta_{v=6}$	β_{OLS}	$\beta_{v=2}$	$\beta_{v=4}$	$\beta_{v=6}$
Number of Omitted Data	4	4	4	4	8	8	8	8
More than 10% Deviation	0	0	0	0	8	0	0	0
5 to 10 % Deviation	2	0	0	0	10	0	0	1
3.0 to 5.0 % Deviation	4	0	0	0	3	2	1	0
2.0 to 3.0	8	2	0	0	2	4	5	8
1.0 to 2.0	9	7	3	0	3	10	11	12
Less than 1.0	7	21	27	30	4	14	13	9
Total	30	30	30	30	30	30	30	30
Maximum Deviation	7.7	2.3	1.2	1.0	12.7	3.1	2.4	2.5
Minimum Deviation	-4.2	-1.5	-0.8	-0.6	-15.2	-4.0	-4.8	-5.7

APPENDIX TABLE A1

Firm	(I) Beta for all data	(II) Beta w/o 4 top data	(III) Beta w/o 8 data	(I)-(II) /std err	(I)-(III) / std err
ALLIED SIGNAL	1.0503	1.0287	0.8909	0.7142	5.2819
	(.030)	(.029)	(.033)		
AMERICAN EXPRESS	1.3902	1.3792	1.3843	0.3528	0.1879
	(.031)	(.032)	(.037)		
A T & T	1.0600	1.0585	1.0665	0.0641	-0.2766
	(.023)	(.024)	(.027)		
BANKAMERICA	1.1381	1.1124	1.0490	0.6372	2.2085
	(.040)	(.041)	(.047)		
CATERPILLAR	0.9912	1.0038	0.9817	-0.4118	0.3115
	(.030)	(.031)	(.035)		
CHEVRON	0.8362	0.8598	0.8339	-0.9413	0.0906
	(.025)	(.026)	(.029)		
COCA-COLA	1.2159	1.1851	1.1823	1.3753	1.4983
	(.022)	(.022)	(.026)		
DEERE	0.9127	0.9293	0.9278	-0.5025	-0.4555
	(.033)	(.034)	(.038)		
DISNEY	1.1811	1.1528	1.0938	0.9389	2.8916
	(.030)	0.0309	0.0351		
DOW CHEMICAL	1.0828	1.0994	1.0999	-0.6600	-0.6807
	(.025)	(.026)	(.029)		
DU PONT	1.0101	1.0400	1.0731	-1.3735	-2.8881
	(.022)	(.022)	(.025)		
KODAK	1.0841	1.0510	0.9529	1.1807	4.6814
	(.028)	(.028)	(.031)		
EXXON	0.9451	0.9221	0.8680	1.0975	3.6807
	(.021)	(.021)	(.024)		
GE	1.1569	1.1595	1.2324	-0.1428	-4.1507
	(.018)	(.019)	(.021)		
GM	1.0557	1.0661	1.0958	-0.3834	-1.4772
	(.027)	(.028)	(.031)		
GOODYEAR	1.0246	1.0053	0.9129	0.5781	3.3347
	(.034)	(.034)	(.039)		
IBM	0.9478	0.9535	0.9207	-0.2458	1.1885
	(.023)	(.023)	(.027)		
INT'L PAPER	1.0966	1.1137	1.0435	-0.6685	2.0753
	(.026)	(.026)	(.029)		
J & J	1.0894	1.1059	1.1555	-0.6804	-2.7193
	(.024)	(.025)	(.028)		
MCDONALDS	1.0039	0.9956	1.0343	0.3434	-1.2695
	(.024)	(.025)	(.028)		
MERCK	0.9263	0.9419	1.0263	-0.6752	-4.3308
	(.023)	(.023)	(.027)		
MMM	1.0127	1.0223	0.9599	-0.5084	2.8060
	(.019)	(.019)	(.022)		
MOBIL	0.9262	0.8874	0.8170	1.5448	4.3431
	(.025)	(.026)	(.029)		
JP MORGAN	1.1958	1.1581	1.0789	1.3914	4.3172
	(.027)	(.026)	(.030)		
PHILIP MORRIS	0.9850	0.9965	1.0738	-0.4706	-3.6300
	(.024)	(.025)	(.028)		
PROCTER & GAMBLE	1.0508	1.0278	0.9900	1.0518	2.7725
	(.022)	(.021)	(.024)		
SEARS	1.1535	1.1383	1.1324	0.5524	0.7697
	(.027)	(.028)	(.032)		
UNION CARBIDE	0.8995	0.9295	1.0091	-0.8195	-2.9952
	(0.037)	(0.036)	(0.041)		
UNITED TECH	0.8385	0.9028	0.9448	-2.3928	-3.9564
	(.027)	(.027)	(.031)		
WOOLWORTH	0.9460	0.9951	1.0494	-1.5971	-3.3616
	(.031)	(.031)	(.036)		

APPENDIX TABLE A2

Portfolio Number	(I) Beta for all data	(II) Beta w/o 4 highest data	(III) Beta w/o 8 data	(I)-(II) over std err	(I)-III) over std err
1	0.4607	0.4598	0.4698	0.06000	-0.59761
	(.015)	(.016)	(.018)		
2	0.5672	0.5639	0.5755	0.28295	-0.70939
	(.012)	(.012)	(.014)		
3	0.6325	0.6327	0.6366	-0.01948	-0.32324
	(.013)	(.013)	(.015)		
4	0.7208	0.7072	0.6785	1.13498	3.52226
	(.012)	(.012)	(.014)		
5	0.8269	0.8460	0.8080	-1.30485	1.28871
	(.015)	(.015)	(.017)		
6	0.8776	0.8825	0.8910	-0.35783	-0.97399
	(.014)	(.014)	(.016)		
7	0.9135	0.9391	0.9649	-1.56990	-3.15718
	(.016)	(.016)	(.018)		
8	0.9365	0.9336	0.9348	0.22552	0.13190
	(.013)	(.013)	(.015)		
9	0.9565	0.9564	0.9265	-0.06964	2.55339
	(.011)	(.011)	(.013)		
10	0.9807	1.0005	0.9703	-1.16677	0.61295
	(.017)	(.017)	(.019)		
11	1.0073	1.0226	1.0240	-1.30895	-1.42779
	(.012)	(.012)	(.013)		
12	1.0423	1.0333	0.9936	0.69029	3.74751
	(.013)	(.013)	(.014)		
13	1.0684	1.0672	1.0406	0.10475	2.35012
	(.012)	(.012)	(.013)		
14	1.0893	1.1145	1.1143	-1.49423	-1.48475
	(.017)	(.017)	(.020)		
15	1.1053	1.1189	1.1313	-0.99743	-1.90759
	(.013)	(.014)	(.016)		
16	1.1257	1.1194	1.0619	0.37876	3.81387
	(.017)	(.017)	(.019)		
17	1.1548	1.1520	1.1274	0.12952	1.24226
	(.022)	(.022)	(.026)		
18	1.2009	1.1973	1.2176	0.27806	-1.30098
	(.013)	(.013)	(.015)		
19	1.2576	1.2801	1.3176	-1.32833	-3.55031
	(.017)	(.017)	(.020)		
20	1.3846	1.4081	1.4867	-1.00132	-4.35127
	(.023)	(.024)	(.027)		